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Isovector Solitons and Maxwell's Equations

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We present an isovector Lagrangian, which admits stable, nonsingular soliton solutions in three space dimensions. The spherical solution and its total energy are obtained via a variational procedure. An antisymmetric, second-rank tensor is defined in terms of the isovector field and its derivatives. This tensor satisfies Maxwell's equations. The corresponding current is identically conserved and the total charge is topologically quantized.

1. INTRODUCTION

Interest in nonlinear field theories possessing soliton solutions is growing. One of the most successful nonlinear relativistic field theories which has proved its physical significance is the model due to Skyrme (1961). In this model, baryons are solitons, having finite radii and self-energies. Extensive work has been done on the phenomenological as well as theoretical aspects of the Skyrme model (see, e.g., Holzwarth and Schwesinger, 1986). Although the predictions of this model are accurate only within ~20%, it is still regarded as a remarkable achievement. Another example of soliton solutions of nonlinear field theories is the magnetic monopole of t'Hooft (1974) and Polyakov (1974). In this case, magnetic monopoles are solitons of a non-Abelian gauge field theory.

In the present paper, we build a Lagrangian density in terms of a three-component scalar field, which undergoes spontaneous symmetry breaking. The corresponding dynamical equations possess stable soliton solutions. Topology of the fields is then used to construct a conserved current which yields quantized charges. The π_2 homotopy group is used for this purpose. Finally, an antisymmetric, second-rank tensor is defined in such a way that

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it satisfies Maxwell's homogeneous and inhomogeneous equations. Despite the apparent similarity with the conventional Maxwell equations, the resulting electrodynamics is highly nonlinear, which is a sort of hidden nonlinearity, as pointed out by Ranada (1990).

2. CHOICE OF THE LAGRANGIAN DENSITY

The elementary field upon which the present work is based is an isovector field $\phi = (\phi_1, \phi_2, \phi_3)$. Each ϕ_a $(a = 1, 2, 3)$ is a pseudoscalar under Lorentz transformations. The Lagrangian density is required to satisfy the following conditions: (1) relativistic covariance, (2) spontaneous breaking of the internal symmetry, (3) stability of the soliton solutions, and (4) finiteness of the energy density.

It can be shown that the simplest choice

$$\mathcal{L} = \frac{1}{2}\,\partial^{\mu}\phi_a\partial_{\mu}\phi_a - V(\phi_a\phi_a) \qquad a = 1, 2, 3 \tag{1}$$

does not satisfy these conditions. The next simplest choice is

$$\mathcal{L} = -\lambda(\partial^{\mu}\phi_a\partial_{\mu}\phi_a)^2 - V(\phi_a\phi_a), \qquad \lambda > 0 \tag{2}$$

where

$$V(\phi_a\phi_a) = b_0\left(1 - \frac{\phi}{\phi_0}\right)^2; \qquad b_0 > 0, \quad \phi = (\phi_a\phi_a)^{1/2} \tag{3}$$

The reason for the negative sign of the kinetic term will become clear shortly [throughout this paper, we use the convention $\eta_{\mu\nu}$= diag(1, −1, −1, −1) for the metric tensor].

The equation of motion derived from (2) reads

$$\partial_{\mu}\partial_{\nu}\phi_b\partial^{\mu}\phi_a\partial^{\nu}\phi_b + \partial_{\mu}\phi_b\partial_{\nu}\partial^{\mu}\phi_b\partial^{\nu}\phi_a + \partial_{\mu}\phi_b\partial^{\mu}\phi_b\Box\phi_a = \frac{1}{\lambda}\frac{\partial V}{\partial\phi_a} \tag{4}$$

The corresponding energy-momentum tensor is

$$T^{\mu\nu} = -4\lambda(\partial^{\alpha}\phi_b\partial_{\alpha}\phi_b)\partial^{\mu}\phi_a\partial^{\nu}\phi_a - \eta^{\mu\nu}\mathcal{L} \tag{5}$$

In the following sections, we will discuss the soliton solutions of (4) and some of their properties.

3. SPHERICAL SOLITONS

We show the existence of spherical solitons, by substituting the ansatz

$$\phi_a = \phi(r)\frac{x^a}{r} \tag{6}$$

into (4) and (5). After straightforward calculation, we obtain

$$3\frac{d^2\phi}{dr^2}\left(\frac{d\phi}{dr}\right)^2 + \frac{2\phi}{r^2}\left(\frac{d\phi}{dr}\right)^2 + \frac{2}{r}\left(\frac{d\phi}{dr}\right)^3 + \frac{2\phi^2}{r^2}\frac{d^2\phi}{dr^2} - \frac{4\phi^3}{r^4} = \frac{1}{4\lambda}\frac{\partial V}{\partial\phi} \tag{7}$$

and

$$\mathcal{H} = T^{00} = \lambda\left[\left(\frac{d\phi}{dr}\right)^4 + 4\frac{\phi^4}{r^4} + 4\frac{\phi^2}{r^2}\left(\frac{d\phi}{dr}\right)^2\right] + V(\phi) \tag{8}$$

The positiveness of all terms in $\mathcal{H}$ is the reason for the negative sign of the kinetic term in (2). The total energy is obviously

$$H = \int_0^\infty \mathcal{H} 4\pi r^4\, dr \tag{9}$$

Equation (7) is also obtainable via minimizing the energy functional (9). The spherical soliton is the solution to (9) subject to the boundary conditions

$$\phi(r) = 0 \quad \text{at} \quad r = 0; \qquad \phi(r) = \phi_0 \quad \text{as} \quad r \to \infty$$

The condition $\phi(0) = 0$ is required to ensure the single-valuedness of ϕ_a at $r = 0$. $\phi(r)$ is expected to become ϕ_0 at infinity, because this value corresponds to the minimum (vacuum) of the potential (3).

The solution to (7) with boundary conditions (10) is obtained through a numerical procedure (variational technique). The following change of variables is very convenient:

$$\psi = \frac{\phi}{\phi_0}; \qquad z = \frac{1}{1 + r/r_0}; \qquad r_0 \equiv \left(\frac{4\lambda}{b_0}\right)^{1/4}\phi_0 \tag{10}$$

r_0 is a constant of dimension length, and determines the scale radius of the soliton. Obviously, $z \in [0, 1]$. This interval is divided into N segments, and a trial function $\psi(z)$ is inserted into the energy integral (11). This trial function could be as simple as the straight line $\psi(z) = 1 - z$. Then $\psi_i = \psi(z_i)$ is varied step by step, derivatives are computed according to the conventional finite-difference expressions, and H is evaluated at each step. The procedure is iterated by a computer program until H is minimized. Convergence is seen to be quite rapid, and the numerical solution is obtained to the desired accuracy.

The asymptotic behavior of the solution to (7) can be obtained by substituting a series ansatz for $\psi(z)$ or $\phi(r)$. The result is

$$\psi(z) = 1 - 2z^4 - 8z^5 + \cdots \qquad \text{for} \quad z \to 0 \tag{11}$$

$$\phi(r) \simeq \phi_0\left(1 - \frac{8\phi_0^4}{b_0 r^4}\cdots\right) \qquad \text{for} \quad r \to \infty \tag{12}$$

$$\phi(r) = k\frac{r}{r_0} - \frac{1}{10k^2}\left(\frac{r}{r_0}\right)^2 + \cdots \tag{13}$$

in which k is a constant fixed by the consistency of the asymptotic solutions. The corresponding total energy is

$$E = 4.05 \cdot 4^{3/4}\pi b_0^{1/4}\phi_0^3 \tag{14}$$

4. TENSOR FORMULATION

Consider the following antisymmetric tensor built from ϕ_a and its derivatives:

$$F^{\alpha\beta} = \frac{4\pi}{c}\epsilon^{\alpha\beta\mu\nu}(\epsilon_{abc}\phi_a\partial_\mu\phi_b\partial_\nu\phi_c + \partial_\nu A_\mu) \tag{15}$$

in which A_μ is a four-vector defined via the wave equation

$$\Box A^\mu - \partial^\mu(\partial_\alpha A^\alpha) = 2\epsilon_{abc}\partial_\alpha(\phi_a\partial^\mu\phi_b\partial^\alpha\phi_c) \tag{16}$$

It can be easily shown that $F^{\mu\nu}$ satisfies Maxwell's equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu \tag{17}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{18}$$

where $\tilde{F}^{\mu\nu}$ is the dual tensor

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \frac{8\pi}{c}\epsilon_{abc}\phi_a\partial^\mu\phi_b\partial^\nu\phi_c + \frac{4\pi}{c}(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{19}$$

and J^ν is the current 4-vector

$$J^\nu \equiv \epsilon^{\mu\nu\alpha\beta}\epsilon_{abc}\partial_\mu\phi_a\partial_\alpha\phi_b\partial_\beta\phi_c \tag{20}$$

This current is conserved,

$$\partial_\nu J^\nu = 0 \tag{21}$$

and the corresponding charge is topologically quantized (Arafune *et al.*, 1973)

$$Q = \int J^0 \, d^3x = ne \tag{22}$$

where $e \equiv 8\pi\phi_0^3$ is the fundamental charge. Topological quantization is fulfilled via the second homotopy group $\pi_2(S^2) = Z$.

It can be easily shown that $n = 1$ for the spherical soliton of Section 3.

If the components of $F^{\mu\nu}$ are identified with the electric and magnetic field components in the usual manner, and J^μ is recognized as the electromagnetic 4-current, we immediately obtain the following results for the spherical soliton:

$$\mathbf{B} = 0 \tag{23}$$

$$\mathbf{E} = \frac{e}{r^2}\hat{r} \qquad \text{as} \quad r \to \infty \tag{24}$$

$$\mathcal{H} = \frac{E^2}{8\pi} \qquad \text{as} \quad r \to \infty \tag{25}$$

$$\mathbf{E} \to 0 \qquad \text{as} \quad r \to 0 \tag{26}$$

The isovector soliton thus resembles a charged particle of finite self-energy and nonsingular electric field.

As the electromagnetic energy density varies as $\sim 1/r^4$ at large radial distances from the charged soliton, we expect inverse-square, Coulombic interaction between two charged solitons at large distances.

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REFERENCES

Arafune, J., Freund, P. G. O., and Goebel, C. J. (1975). *Journal of Mathematical Physics*, **16**(2), 433.

Holzwarth, G., and Schwesinger, B. (1986). *Reports on Progress in Physics*, **49**, 825.

Polyakov, A. M. (1974). *JETP Letters*, **20**, 194.

Ranada, A. F. (1990). In *Solitons and Applications, IVth International Workshop (Dubna, VSSR)*, World Scientific, Singapore.

Riazi, N. (n.d.). Isovector electrodynamics with quantized electric charges, preprint.

Skyrme, T. H. R. (1961). *Proceedings of the Royal Society A*, **260**, 127.

T'Hooft, G. (1974). *Nuclear Physics B*, **79**, 276.